Eigen Decomposition

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Eigenvectors and EigenValues

Suppose A is a square matrix.

An Eigenvector is any vector (x), such that multiplying it by A returns a scaled version of itself. The scalar is the Eigenvalue (λ).

$$Ax = \lambda x \qquad [0.1]$$

*Of course, where $x \neq 0$.

Example.

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \qquad [0.2]$$

From our definition:

$$Ax = \lambda x \qquad [0.3]$$

Rewrite by inserting an identity matrix (I).

$$Ax = \lambda I x \qquad [0.4]$$

Subtract RHS to LHS. Factor out X.

$$\begin{aligned} Ax - \lambda I x &= 0 \\ (A - \lambda I)x &= 0 \end{aligned} \qquad [0.5]$$

Suppose we let M represent $(A - \lambda I)$. From our original definition, we're saying that M contains some vector (x) which isn't equal to 0, but produces 0. This also means that M cannot be invertible, because: in addition to the 0 vector mapping to 0, there is also another vector that maps to 0.

We also know that if a matrix is not invertible, it's determinant equals 0. Thus, M's determinant must be 0.

$$det(M) = |M| = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc = 0 \qquad [0.6]$$

Thus:

$$det(A - \lambda I) = 0 \qquad [0.7]$$

Rewriting in full form:

$$\begin{aligned} det\left(\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) &= 0 \\ det\left(\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) &= 0 \\ det\left(\begin{bmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix}\right) &= 0 \end{aligned} \quad [0.8]$$

Now we want to know the determinant of the last matrix (M). From our equation for the determinant we get:

$$\begin{aligned} det\left(\begin{bmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix}\right) &= (3\lambda + \lambda^2) - (-2) \\ &= 3\lambda + \lambda^2 + 2 \end{aligned} \quad [0.9]$$

We have our final polynomial form equal to 0, which we can factor.

$$\begin{aligned} 3\lambda + \lambda^2 + 2 &= 0 \\ (\lambda + 2)(\lambda + 1) &= 0 \end{aligned} \quad [0.10]$$

Resulting in our two Eigenvalues:

$$\lambda = -2, \lambda = -1 \quad [0.11]$$

To find the corresponding EigenVectors, we need to substitute back in our values for lambda, individually.

$$\begin{aligned} Ax &= \lambda x \\ \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= -2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \begin{bmatrix} x_2 \\ -2x_1 - 3x_2 \end{bmatrix} &= \begin{bmatrix} -2x_1 \\ -2x_2 \end{bmatrix} \end{aligned} \quad [0.12]$$

Which we can solve for the vectors.

Steps:

1. Set equation equal to 0
2. Solve for Eigenvalues
3. Insert Eigenvalues to solve for Eigenvectors